

# The Optical Mechanics of Gravity

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*Dedicated with admiration, gratitude and deep respect  
to Professor Gary Gibbons on his 70th birthday*

## Abstract

We have studied optical metrics via null geodesics, formulated classical mechanics in optical-mechanical terms, and described the geometry of mechanical systems with drag. Then, we apply the formulation to other solutions of Einstein's equations in spherically symmetric spaces and deduce the related Binet's equation. Finally, we review the dualities between different systems arising from conformal transformations that preserve the Jacobi metric.

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# 1 Introduction

Optical metrics are essentially null geodesics in a given spacetime. Such null arcs can be studied by projecting the curves onto lower dimensional spatial surfaces. As example, if a metric admits a timelike Killing vector  $K$  orthogonal to a hypersurface, the null geodesic will project down to unparameterised geodesics of the optical metric on the space of orbits of  $K$ . Similar constructions were studied for metrics admitting a stationary Killing vector [1] or a timelike conformal retraction [2] where the projected null curves provide some notion of geometric structure to a hypersurface. Because the null metric vanishes, the geodesics are defined by minimising only the spatial part of the metric.

Geodesics on null curves are formulated in accordance with Fermat's principle, as the spacetime curves cannot be minimized. One direct utility is in observational astronomy in the study of gravitational lensing. Using null geodesics, one can interpret gravitational fields as transparent media with a refractive index. Conversely, one can also view transparent media as regions with localised gravitational fields, as speculated by P. de Fermat and P.L. Maupertuis [3]. Furthermore, from the Maupertuis form of the action integral and its Lagrangian, one can say that in the classical limit, the resulting geodesic traversed by a body is a null geodesic.

Since we are discussing null-geodesic formulation using only the spatial part of the metric, we must also consider the Jacobi metric [5, 6, 7], which deals with a similar concept of reducing the geodesic metric, usually stationary ones to just the spatial part. The analytical calculations involving null geodesics in spherically symmetric spacetimes using Weierstrass elliptic functions are given in [8]. Demanding preservation of the classical Jacobi metric under conformal transformation can lead us to interesting dual pairs of mechanical systems, such as the Bohlin transformation [10, 11], as shown by Casey in [12]. These dual systems can be written in the form of Binet's equations, casting them as central force solution for particular potentials. We also briefly mention the anisotropic case, the optical anisotropy of curved space is demonstrated by means of a rigorous algebraic analysis in [9].

Most practical mechanical problems are required to deal with drag effects that will occur during their operation. Performing an optical mechanical formulation for such systems helps generalize our analysis. The metric related to such mechanical systems should describe a more general category of spacetimes.

In this article, we shall first review the preliminaries on null geodesics, derive Snell's Law used in refractive optics, and describe the refractive indices of the Kerr metric. Then we shall demonstrate how isotropic spaces produce mechanics involving drag, and how to deduce the spacetime metric from a damped equation of motion from in the optical-mechanical form. We shall also write the Jacobi metric, both relativistic and non-relativistic in optical-mechanical form using refractive indices.

In the next section, we shall extrapolate S. Casey's results for  $n$ -dimensional Schwarzschild metrics in [12] to the general spherically symmetric metric. We shall first formulate the optical metric for spherically symmetric metrics in general and deduce the related equations and dynamics that describe such geodesics. We shall then proceed to write the Binet's equations related to null geodesics to see if we can describe it as a central force solution in two dimensions, deduce a potential, and apply the analysis to two other examples of solutions to Einstein's equations besides the Schwarzschild metric.

Following that, we shall explore the various mechanical dualities that arise from preservation of the classical Jacobi metric under a conformal co-ordinate map, one example being Bohlin's transformation.

## 2 Preliminaries

It was speculated P.L. Maupertuis in [3] how the refraction of light upon passing into a medium could be due to gravitational effects. Here, we will demonstrate with a null geodesic in isotropic space how refractive phenomena can arise from a gravitational metric, as shown in [13].

Suppose we have an isotropic space-time metric given by:

$$ds^2 = A(\vec{r})c^2 dt^2 - B(\vec{r})|d\vec{r}|^2 \quad \Rightarrow \quad \left(\frac{ds}{d\tau}\right)^2 = A(\vec{r})c^2 \dot{t}^2 - B(\vec{r})|\dot{\vec{r}}|^2 \quad (2.1)$$

Now, if we take the null geodesic equation of (2.1), we will have the equation:

$$ds^2 = 0 \quad \Rightarrow \quad |\dot{\vec{r}}|_{null}^2 = \frac{A(\vec{r})}{B(\vec{r})}c^2 \dot{t}^2 \quad \Rightarrow \quad \frac{B(\vec{r})}{A(\vec{r})}v_{null}^2 = c^2 \quad v = \frac{|\dot{\vec{r}}|}{\dot{t}}$$

One could say that the local refractive index wrt vaccum is given by

$$n(\vec{r}) = \frac{c}{v_{null}} = \sqrt{\frac{B(\vec{r})}{A(\vec{r})}} \quad (2.2)$$

The solutions to Einstein's equations for vaccum usually have  $AB = 1$ . Under this condition, the refractive index (2.2) becomes:

$$n(\vec{r}) = \frac{1}{A(\vec{r})} \quad (2.3)$$

If we are dealing with an anisotropic space, then we will have different refractive indices for each individual spatial direction given by  $n_i(\vec{r})$ . Here the stationary metric, where  $g_{ij}$  is assumed to have been diagonalised via similarity tranformation, is written as

$$ds^2 = g_{00}(\vec{r})c^2 dt^2 - 2g_{i0}(\vec{r})c dt dx^i - g_{ij}(\vec{r})dx^i dx^j \quad (2.4)$$

where the metric components given above  $g_{00}, g_{ij} > 0 \forall i, j$  are posetive definite. The refractive index along each direction can be deduced by setting all other co-ordinates constant at a time.

$$\begin{aligned} ds^2 = 0, \quad x^j = constant \quad \forall \quad j \neq i \\ \Rightarrow \quad g_{00}(\vec{r})c^2 - 2g_{i0}(\vec{r})c v_{null}^i - g_{ii}(\vec{r})(v_{null}^i)^2 = 0 \quad v^i = \frac{\dot{x}^i}{\dot{t}} \end{aligned} \quad (2.5)$$

If we choose to write (2.5) as a quadratic equation for thte refractive index  $n_i$ , we will have:

$$g_{00}(\vec{r})(n_i)^2 - 2g_{i0}(\vec{r})n_i - g_{ii}(\vec{r}) = 0 \quad n_i = \frac{c}{v_{null}^i} \quad (2.6)$$

with the two solutions of (2.6):

$$n_i = \frac{g_{i0} \pm \sqrt{(g_{i0})^2 + g_{00}g_{ii}}}{g_{00}} \quad (2.7)$$

Now, it is clearly evident that if  $g_{ii}$  and  $g_{00}$  are posetive definite, then

$$(g_{i0})^2 + g_{00}g_{ii} > (g_{i0})^2 \quad \Rightarrow \quad \sqrt{(g_{i0})^2 + g_{00}g_{ii}} > |g_{i0}|$$

$$\Rightarrow \sqrt{(g_{i0})^2 + g_{00}g_{ii}} - |g_{i0}| > 0 \quad (2.8)$$

Now we may consider freedom of the signature of  $g_{i0}$ , depending on the vector potential applied within the space (2.4) such that

$$g_{i0} = \pm |g_{i0}|$$

However, regardless of its signature, if we choose the  $-$  sign option in (2.7), we shall have a negative refractive index according to (2.8), since:

$$n_i = \begin{cases} \frac{|g_{i0}| - \sqrt{(g_{i0})^2 + g_{00}g_{ii}}}{g_{00}} < 0 & ; \quad g_{i0} = |g_{i0}| \\ -\frac{|g_{i0}| + \sqrt{(g_{i0})^2 + g_{00}g_{ii}}}{g_{00}} < 0 & ; \quad g_{i0} = -|g_{i0}| \end{cases}$$

which would technically imply that light is travelling in a direction opposite to the direction it would take in vacuum. Since we wish to consider the realistic solution, we shall only take the  $+$  sign option in (2.7), which means:

$$n_i = \frac{g_{i0} + \sqrt{(g_{i0})^2 + g_{00}g_{ii}}}{g_{00}} \quad (2.9)$$

It must be noted that considering negative refractive index is permissible when considering metamaterials [14, 15, 16]. Now we should be able to describe a mechanical formulation of isotropic spacetimes using this optical interpretation of the metric.

## 2.1 Dynamics of null-geodesics

Since we have constrained the length of a null geodesic to vanish, applying a variational process upon its length seems futile. It is more sensible to vary the spatial part alone, effectively applying Fermat's principle of light travelling by the shortest path between two points. Thus, using (2.2) we can say that the optical arc integral, and its Euler-Lagrange equation are

$$\begin{aligned} l = \int_1^2 d\tau \, c\dot{t} &= \int_1^2 d\tau \, n(\vec{r})|\dot{\vec{r}}| & L = c\dot{t} &= n(\vec{r})|\dot{\vec{r}}| \\ \delta l = 0 &\Rightarrow \frac{d}{d\tau} \left( n(\vec{r}) \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \right) &= |\dot{\vec{r}}| \vec{\nabla} n(\vec{r}) \end{aligned} \quad (2.1.1)$$

We wish to regard the arc length as a natural parameter along the curve. If we define a new parametrization with respect to arc length, the reparametrized velocity can be written as a unit vector denoting direction of the light ray.

$$\frac{d}{d\sigma} = \frac{1}{|\dot{\vec{r}}|} \frac{d}{d\tau} \quad \Rightarrow \quad \hat{e} = \frac{d\vec{r}}{d\sigma} = \frac{\dot{\vec{r}}}{|\dot{\vec{r}}|} \quad , \quad |\hat{e}|^2 = 1 \quad (2.1.2)$$

which lets us write the Maupertuis Lagrangian for light-like null curves as:

$$\begin{aligned} \vec{p}_{null} &= \frac{\partial L}{\partial \dot{\vec{r}}} = n(\vec{r}) \hat{e} \\ \therefore L_{null} &= n(\vec{r}) \hat{e} \cdot \dot{\vec{r}} = n(\vec{r}) |\dot{\vec{r}}| \quad \Rightarrow \quad l = \int_1^2 n(\vec{r}) |d\vec{r}| \end{aligned}$$

Since the null geodesic path integral is given as shown above, we can conclude as shown in [6] that for the geodesic time integral  $T = \int_1^2 d\tau \dot{t} = \frac{1}{c} \int_1^2 dl$

$$\frac{\partial T}{\partial \vec{x}} = \frac{1}{c} \frac{\partial l}{\partial \vec{x}} = \frac{1}{c} \frac{\partial L}{\partial \dot{\vec{x}}} = \frac{\vec{p}}{c} = \frac{n(\vec{r})}{c} \hat{e}$$

which leads us to the Eikonal equation:

$$\left| \frac{\partial T}{\partial \vec{x}} \right|^2 = \left( \frac{\partial T}{\partial x} \right)^2 + \left( \frac{\partial T}{\partial y} \right)^2 + \left( \frac{\partial T}{\partial z} \right)^2 = \frac{n^2}{c^2} \quad (2.1.3)$$

Furthermore, we can use (2.1.2) to rewrite the Euler-Lagrange equation (2.1.1) and derive from it a result [13] normally derived from the Eikonal equation:

$$\frac{d}{d\sigma} \left( n(\vec{r}) \hat{e} \right) = \vec{\nabla} n(\vec{r}) \quad (2.1.4)$$

$$\begin{aligned} \Rightarrow \quad n(\vec{r}) \frac{d\hat{e}}{d\sigma} + (\hat{e} \cdot \vec{\nabla} n) \hat{e} &= (\hat{e} \cdot \hat{e}) \vec{\nabla} n(\vec{r}) \\ \Rightarrow \quad n(\vec{r}) \frac{d\hat{e}}{d\sigma} &= (\hat{e} \cdot \hat{e}) \vec{\nabla} n(\vec{r}) - (\hat{e} \cdot \vec{\nabla} n) \hat{e} = (\hat{e} \times \vec{\nabla} n(\vec{r})) \times \hat{e} \\ &\Rightarrow \quad \frac{d\hat{e}}{d\sigma} = (\hat{e} \times \vec{\nabla} \ln n) \times \hat{e} \end{aligned} \quad (2.1.5)$$

These spatial geodesics that such equations describe are better analysed by using the orthonormal frame of the Frenet-Serret formalism [17]. We will use this result to demonstrate an Snell's law for refractive optics is applicable to gravitational fields as well.

### 2.1.1 Snell's Law of refraction

Now choose a basis in two dimensions  $(\hat{n}_{\parallel}, \hat{n}_{\perp})$  set up around the direction of  $\vec{\nabla} n(\vec{r})$ , where  $\hat{n}_{\parallel}$  denotes direction along  $\vec{\nabla} n(\vec{r})$ , while  $\hat{n}_{\perp}$  denotes direction orthogonal to it. We can therefore write for unit vector and derivative of refractive index:

$$\hat{e} = \cos \theta \hat{n}_{\parallel} + \sin \theta \hat{n}_{\perp} \quad \frac{dn}{d\sigma} = \frac{d\vec{r}}{d\sigma} \cdot \vec{\nabla} n = \hat{e} \cdot \vec{\nabla} n = |\vec{\nabla} n| \cos \theta$$

Applying the above equations to (2.1.5) gives us the LHS and RHS as follows, resulting in a solvable differential equation:

$$\begin{aligned} \left. \begin{aligned} \frac{d\hat{e}}{d\sigma} &= \left( -\sin \theta \hat{n}_{\parallel} + \cos \theta \hat{n}_{\perp} \right) \frac{d\theta}{d\sigma} \\ (\hat{e} \times \vec{\nabla} n) \times \hat{e} &= \sin \theta \left( \sin \theta \hat{n}_{\parallel} - \cos \theta \hat{n}_{\perp} \right) |\vec{\nabla} n| \end{aligned} \right\} \Rightarrow -n(\vec{r}) \sin \theta \frac{d\theta}{d\sigma} = \sin^2 \theta |\vec{\nabla} n| = \frac{\sin^2 \theta}{\cos \theta} \frac{dn}{d\sigma} \\ \Rightarrow \quad \sin \theta \, dn + n \cos \theta \, d\theta &= d(n \sin \theta) = 0 \end{aligned}$$

Thus we have the conserved quantity of null-geodesic dynamics

$$n(\vec{r}) \sin \theta = \text{const} \quad (2.1.6)$$

which is Snell's Law from refractive optics. This supports the theory that regions with gravitational fields can be regarded as refractive media, and vice versa.

### 2.1.2 Application to the Kerr metric

The black-hole spacetime known as the rotating (Kerr) black hole is a stationary metric.

The Kerr metric (setting  $c = 1$ ) is:

$$ds^2 = \left(1 - \frac{2GMr}{\rho^2}\right) dt^2 - \frac{4GMar \sin^2 \theta}{\rho^2} d\phi dt - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta \right] d\phi^2, \quad (2.1.7)$$

$$\Delta(r) = r^2 - 2GMr + a^2 \quad \rho^2(r, \theta) = r^2 + a^2 \cos^2 \theta.$$

Using the formulation for non-isotropic spaces (2.4) - (2.9), we can say that it has the following refractive indices:

$$\begin{aligned} n_r &= \frac{\rho^2}{\sqrt{\Delta(\rho^2 - 2GMr)}} \\ n_\theta &= \frac{\rho^2}{\sqrt{\rho^2 - 2GMr}} \\ n_\phi &= \frac{4GMar \sin^2 \theta}{\rho^2 - 2GMr} + \sin \theta \sqrt{\left( \frac{2GMar \sin \theta}{\rho^2 - 2GMr} \right)^2 + \frac{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta}{\rho^2 - 2GMr}} \end{aligned} \quad (2.1.8)$$

Now, we will show how to optically describe classical particle mechanics, or the optical mechanical formulation.

## 2.2 Optical-mechanical formulation

Using (2.3) we will proceed to rewrite the metric (2.1) in as conventional a form as possible, and deduce the form of the non-relativistic Lagrangian. A metric that is a solution to Einstein's equations for vacuum ( $AB = 1$ ) can be written as:

$$ds^2 = \frac{c^2 dt^2}{n(\vec{r})} - n(\vec{r}) |d\vec{r}|^2 = c^2 dt^2 \left[ \frac{1}{n(\vec{r})} - n(\vec{r}) \left( \frac{|\vec{v}|}{c} \right)^2 \right] \quad (2.2.1)$$

If  $A(\vec{r}) = \frac{1}{n(\vec{r})} = 1 + \frac{2U(\vec{r})}{mc^2}$ , we can say that:

$$ds^2 = c^2 dt^2 \left[ 1 - \frac{2}{mc^2} \left\{ n(\vec{r}) \left( \frac{1}{2} m |\vec{v}|^2 \right) - U(\vec{r}) \right\} \right]$$

Starting from this metric, we shall describe the optical-mechanical formulation of classical mechanics on such spaces.

### 2.2.1 Classical motion with drag

If keeping in mind that  $T = \frac{1}{2}mn(\vec{r})|\vec{v}|^2$ , where  $mn(\vec{r})$  acts effectively as position-dependent mass, we can write the non-relativistic Lagrangian and energy via Legendre's transformation for this space and the classical form of equations of motion:

$$\begin{aligned} L_{opt} &= \left( \frac{1}{2}mn(\vec{r})|\vec{v}|^2 \right) - U(\vec{r}) = T - U(\vec{r}) \\ \vec{p} &= \frac{\partial L_{opt}}{\partial \vec{v}} \quad \Rightarrow \quad E = \vec{p} \cdot \vec{v} - L_{opt} = T + U \\ m \frac{d}{dt} (n(\vec{r})\vec{v}) &= -\vec{\nabla} U. \end{aligned} \quad (2.2.2)$$

We shall define the rest Lagrangian  $L_0$  and rest energy  $E_0$ , to better write the relativistic Lagrangian as:

$$L_0 = -E_0 = -mc^2. \quad (2.2.3)$$

Since  $U = \frac{E_0}{2} \left( \frac{1}{n(\vec{r})} - 1 \right)$ , then the equation (2.2.2) is given by

$$mn(\vec{r})\ddot{\vec{r}} + m(\dot{\vec{r}} \cdot \vec{\nabla} n(\vec{r}))\dot{\vec{r}} + \frac{E_0}{2} \vec{\nabla} \left( \frac{1}{n(\vec{r})} \right) = 0. \quad (2.2.4)$$

This equation can be further reduced to

$$\ddot{\vec{r}} + \frac{\dot{\vec{r}} \cdot \vec{\nabla} n(\vec{r})}{n(\vec{r})} \dot{\vec{r}} - \frac{\mathcal{E}_0}{2} \frac{\vec{\nabla} n(\vec{r})}{(n(\vec{r}))^3} = 0, \quad \text{where } \mathcal{E}_0 = \frac{E_0}{m} = c^2. \quad (2.2.5)$$

which is effectively comparable to a Gorriange-Leach mechanical system with damping quadratically dependent on velocity. This shows that a spacetime that is conformally flat in the spatial part with non-unity conformal factor will exhibit some form of viscous drag. We can write the optical-mechanical relativistic Lagrangian as follows:

$$\mathcal{L} = -mc \left| \frac{ds}{dt} \right| = -mc^2 \sqrt{1 - \frac{2L_{opt}}{mc^2}} = L_0 \sqrt{1 + 2 \frac{L_{opt}}{L_0}}.$$

In the classical limit  $\frac{L_{opt}}{L_0} \ll 1$ , we have:

$$\begin{aligned} \mathcal{L} \xrightarrow{L_{opt} \ll mc^2} L &= L_0 + L_{opt} = \left( \frac{1}{2}mn(\vec{r})|\vec{v}|^2 \right) - U(\vec{r}) + L_0 \\ &= \frac{mn(\vec{r})}{2} |\vec{v}|^2 - \frac{mc^2}{2} \left( 1 + \frac{2U(\vec{r})}{mc^2} \right) - \frac{mc^2}{2} = \frac{m}{2} \left[ n(\vec{r})|\vec{v}|^2 - \frac{c^2}{n(\vec{r})} \right] + \frac{L_0}{2} \end{aligned}$$

Omitting the additive constant  $\frac{L_0}{2}$ , the effective classical Lagrangian parametrized wrt  $\tau$  is:

$$L_{class} \approx -\frac{m}{2} \left( \frac{ds}{d\tau} \right)^2 = \frac{m}{2} n(\vec{r}) |\dot{\vec{r}}|^2 - \frac{E_0 t^2}{2n(\vec{r})} \quad \Rightarrow \quad \begin{cases} \vec{p} = \frac{\partial L_{class}}{\partial \dot{\vec{r}}} = mn(\vec{r})\dot{\vec{r}} \\ p_t = -\frac{\partial L_{class}}{\partial t} = -\frac{E_0}{n(\vec{r})} t = -E \end{cases} \quad (2.2.6)$$

Both,  $L_{opt}$  and  $L_{class}$  can be applied for the purpose of deducing the classical equations of motion since the difference is an additive term of  $\frac{L_0}{2}$ .

### 2.2.2 Jacobi-Maupertuis description

In Maupertuis form [6], the geodesic action integral can be described using the Maupertuis Lagrangian  $L_{Maup}$ , which is more or less the effective Lagrangian  $L_{eff}$  as in [19]

$$L_{Maup} = p_\mu \dot{x}^\mu = \vec{p} \cdot \dot{\vec{r}} + p_t \dot{t} \quad (2.2.7)$$

$$S = \int_1^2 d\tau L_{Maup} = \int_1^2 (\vec{p} \cdot d\vec{r} + p_t dt) \quad \Rightarrow \quad \frac{\partial S}{\partial \vec{r}} = \frac{\partial L_{class}}{\partial \dot{\vec{r}}} = \vec{p} = mn(\vec{r}) \dot{\vec{r}} \quad (2.2.8)$$

From (2.2.7), we can see that the overall geodesic Hamiltonian vanishes, from which, we realize that on applying (2.2.6), the classical geodesic Lagrangian is effectively null

$$(L_{class})_{along\ the\ geodesic} = L_{Maup}$$

$$\therefore p_\mu \dot{x}^\mu - L_{Maup} = p_\mu \dot{x}^\mu - L_{class} = \frac{1}{2mn(\vec{r})} |\vec{p}|^2 - \frac{n(\vec{r})}{2mc^2} E^2 = 0 \quad (2.2.9)$$

This means that the classical Hamilton-Jacobi equation with (2.2.8) gives us

$$E = \vec{p} \cdot \vec{v} - L_{opt} = \frac{|\vec{p}|^2}{2mn(\vec{r})} + U(\vec{r}) \quad \Rightarrow \quad |\vec{p}|^2 = 2mn(\vec{r})(E - U(\vec{r}))$$

$$\vec{p} = \frac{\partial S}{\partial \vec{r}} \quad \Rightarrow \quad \left| \frac{\partial S}{\partial \vec{r}} \right|^2 = 2mn(\vec{r})(E - U(\vec{r})) \quad (2.2.10)$$

Furthermore, from (2.2.9) and (2.2.10), we can say that:

$$|\vec{p}|^2 = 2mn(\vec{r})(E - U(\vec{r})) = \left( \frac{n(\vec{r})E}{c} \right)^2 \quad \Rightarrow \quad E - U(\vec{r}) = \frac{E^2 n(\vec{r})}{2mc^2} \quad (2.2.11)$$

which is in contrast with what is stated in [19, 20]. Now, since the metric is time-independent, the momentum conjugate to time is constant  $p_t \approx -E$ , so we can say that the effective Lagrangian is given from (2.2.8) by:

$$S = \int_1^2 dt (\vec{p} \cdot \vec{v} - E)$$

$$\delta S = \delta \int_1^2 dt \vec{p} \cdot \vec{v} = 0 \quad \Rightarrow \quad L_{eff} = \vec{p} \cdot \vec{v} = 2n(\vec{r})T \quad \vec{p} = \frac{\partial L_{opt}}{\partial \vec{v}} = mn(\vec{r})\vec{v}$$

showing that the effective action covers only the spatial part of the geodesic.

$$L_{eff} = \left| \frac{ds}{d\tau} \right| = \sqrt{n(\vec{r})T} \sqrt{4n(\vec{r})T} = \sqrt{E - U(\vec{r})} \sqrt{2mn(\vec{r})|\vec{v}|^2}$$

and according to [6], using (2.2.11) the classical Jacobi metric is given by:

$$ds_{cJ}^2 = 2mn(\vec{r})(E - U(\vec{r}))|d\vec{r}|^2 = \left( \frac{En(\vec{r})}{c} \right)^2 |d\vec{r}|^2 \quad (2.2.12)$$

and the relativistic Jacobi metric according to [5, 6], is given by:

$$ds_{rJ}^2 = \left( \frac{\mathcal{E}^2}{c^2} n(\vec{r}) - m^2 c^2 \right) n(\vec{r}) |d\vec{r}|^2 \quad (2.2.13)$$

Now we shall elaborate on the geometric formulation of damped mechanical systems.



## 2.3 Equations with drag term and related geometry

We have seen in (2.2.4) that motion through a spacetime described by the metric (2.2.1) is influenced by drag. The best way to analyze and solve it is by comparison to a generalized mechanical systems influenced by drag such as the Gorriange Leach equation. We shall briefly review the Gorriange Leach equation and see how it helps us analyze and solve (2.2.4). Such systems must be carefully examined as they may or may not be integrable. We shall investigate some integrable cases of mechanical systems with drag terms.

For the general form of a metric with only a scalar potential

$$ds^2 = \left(1 + \frac{2U}{mc^2}\right) c^2 dt^2 - |d\vec{x}|^2 = A(\vec{x}) c^2 dt^2 - |d\vec{x}|^2, \quad A(\vec{x}) = 1 + \frac{2U}{mc^2}. \quad (2.3.1)$$

and its corresponding classical Lagrangian

$$L_{class} = -\frac{m}{2} \left( \frac{ds}{dt} \right)^2 = \frac{m}{2} |\vec{x}'|^2 - \frac{mc^2}{2} A(\vec{x}).$$

we shall get the following equations of motion:

$$\Rightarrow \quad \vec{x}'' = -\frac{c^2}{2} \vec{\nabla} A(\vec{x}) \equiv -\frac{1}{m} \vec{\nabla} U(\vec{x}) \quad |\vec{v}|^2 = g_{ij}(\vec{x}) v^i v^j \quad (2.3.2)$$

Now if we consider the damped equation of motion given by

$$\ddot{\vec{x}} + h(\vec{x}; \dot{\vec{x}}) \dot{\vec{x}} + \frac{1}{m} \vec{\nabla} U = 0 \quad (2.3.3)$$

which can be re-written into

$$\ddot{\vec{x}} + h(\vec{x}; \dot{\vec{x}}) \dot{\vec{x}} + \frac{c^2}{2} \vec{\nabla} \left( 1 + \frac{2U}{mc^2} \right) = 0 \quad (2.3.4)$$

Reparametrization  $t \longrightarrow \tau = \tau(t)$  to convert (2.3.4) to a more suitable form gives:

$$\vec{x}'' + \frac{\ddot{\tau} + \dot{\tau} h}{\dot{\tau}^2} \vec{x}' + \frac{1}{m \dot{\tau}^2} \vec{\nabla} U = 0 \quad (2.3.5)$$

Under the special circumstances:

$$\begin{aligned} \ddot{\tau} + \dot{\tau} h = 0 & \Rightarrow \quad \dot{\tau} = \dot{\tau}_0 e^{-\int dt \cdot h} = e^{-\int dt \cdot h} \quad (\because \dot{\tau}_0 = 1) \\ \text{and} \quad h(\vec{x}; \dot{\vec{x}}) = \dot{\alpha}(\vec{x}, t) & \Rightarrow \quad \dot{\tau} = e^{-\alpha(\vec{x}, t)} \end{aligned} \quad (2.3.6)$$

we transform (2.3.5) into a more familiar form of equation of motion:

$$\vec{x}'' = -\frac{e^{2\alpha(\vec{x}, t)}}{m} \vec{\nabla} U = -\frac{c^2 e^{2\alpha(\vec{x}, t)}}{2} \vec{\nabla} \left( 1 + \frac{2U}{mc^2} \right) \quad (2.3.7)$$

Thus, upon comparing (2.3.7) to (2.3.2), we will have the corresponding classical Lagrangian:

$$\vec{x}'' = -\frac{e^{2\alpha}}{m} \vec{\nabla} U \quad \longrightarrow \quad \tilde{L}_d = \frac{m}{2} \left( |\vec{x}'|^2 - e^{2\alpha} c^2 A(\vec{x}) \right)$$

then using (2.3.6), we have  $d\tau = e^{-\alpha(\vec{x},t)} dt$ , and in the classical limit, we will have the classical action  $S_{class}$  given as

$$\begin{aligned}\tilde{L}_d &= -\frac{m}{2} \left( \frac{ds}{d\tau} \right)^2 = \frac{m}{2} \left( |\dot{\vec{x}}|^2 - c^2 e^{2\alpha} A(\vec{x}) \right) = e^{2\alpha} \frac{m}{2} \left( |\dot{\vec{x}}|^2 - c^2 A(\vec{x}) \right) \\ dS_{class} &= \tilde{L}_d d\tau = d\tau e^{2\alpha} \left( \frac{m}{2} |\dot{\vec{x}}|^2 - \frac{mc^2}{2} A \right) = dt e^{\alpha} \left( \frac{m}{2} |\dot{\vec{x}}|^2 - \frac{mc^2}{2} A \right) = L_d dt \\ \Rightarrow \quad L_d &= e^{-\alpha} \tilde{L}_d = e^{\alpha} \left( \frac{m}{2} |\dot{\vec{x}}|^2 - \frac{mc^2}{2} A \right)\end{aligned}$$

Thus, the damped Lagrangian can be written by comparing (2.3.2) to (2.3.7) as:

$$L_d = -\frac{m}{2} \left( \frac{ds}{dt} \right)^2 = e^{\alpha(\vec{x},t)} \left( \frac{m}{2} |\dot{\vec{x}}|^2 - \frac{mc^2}{2} A \right) \quad (2.3.8)$$

and the metric corresponding to (2.3.3) just as (2.3.1) corresponds (2.3.2) to on writing will be:

$$\begin{aligned}ds^2 &= -\frac{2}{m} L_d dt^2 = e^{\alpha(\vec{x},t)} \left( A(\vec{x}) c^2 dt^2 - |d\vec{x}|^2 \right) \\ ds^2 &= e^{\alpha(\vec{x},t)} \left[ \left( 1 + \frac{2U}{mc^2} \right) c^2 dt^2 - |d\vec{x}|^2 \right] \quad (2.3.9)\end{aligned}$$

Comparing (2.3.4) to (2.2.5), we have  $\alpha = \ln n(\vec{r})$ , and  $\frac{c^2}{2} \left( 1 + \frac{2U}{mc^2} \right) = \frac{\mathcal{E}_0}{2(n(\vec{r}))^2}$ , so the metric by reverse calculation will be

$$ds^2 = \frac{c^2 dt^2}{n(\vec{r})} - n(\vec{r}) |d\vec{x}|^2$$

reproducing the metric (2.2.1). Thus, we have obtained the form of the metric and the Lagrangian with damping effects applied by converting it into an undamped form via suitable reparametrization. Conversely, undamped systems can also be described as damped systems via suitable reparametrization.

### 2.3.1 The Gorriange Leach equation

In 1993, Gorriange and Leach exhibited two classes of differential equations incorporating drag terms while having closed elliptical orbits

$$\ddot{z} + h(z, \bar{z}; \dot{z}, \dot{\bar{z}}) \dot{z} + g(z, \bar{z}) z = 0 \quad (2.3.10)$$

we transform (2.3.10) into the form of (2.3.7):

$$z'' + g(z, \bar{z}) e^{2\alpha(|z|,t)} z = 0$$

For a damped spherically symmetric Harmonic Oscillator, we have under this parametrization:

$$z'' + \omega^2 z = 0$$

if we write  $H(z, \bar{z}) = \alpha(r)$ , then we have:

$$\begin{aligned} g(z, \bar{z})e^{2\alpha(|z|,t)}z &= e^{2\alpha(|z|,t)}\frac{\partial U(|z|)}{\partial \bar{z}} = \omega^2 z \\ \Rightarrow \quad \frac{\partial U(|z|)}{\partial \bar{z}} &= \omega^2 e^{-2\alpha(r,t)}z \quad \Rightarrow \quad U(|z|) = e^{-2\alpha(r,t)}\frac{\omega^2}{2}|z|^2 \end{aligned} \quad (2.3.11)$$

Working backwards from the Euler-Lagrange equation of motion and using (2.3.11), we can say that according to (2.3.8), we can write the Lagrangian as in [28]:

$$L = \frac{m}{2}e^{-\alpha(|z|,t)}(|\dot{z}|^2 - \omega^2|z|^2)$$

and the metric for such a dissipative system (2.3.10) according to (2.3.9) is:

$$ds^2 = e^{-\alpha(|z|,t)}\left[\left(1 + \frac{\omega^2|z|^2}{c^2}\right)c^2 dt^2 - |dz|^2\right]$$

Thus, we have the metric and the Lagrangian for the Gorriange Leach equation.

### 2.3.2 Damped Kepler-Hooke duality

Therefore, using (2.3.6) and (2.3.11), (2.3.10) will lead to:

$$\ddot{z} + \dot{\alpha}\dot{z} + \omega^2 e^{-2\alpha(r,t)}z = 0 \quad (2.3.12)$$

$$\begin{aligned} \Rightarrow \quad 2(\dot{z}\ddot{z} + \dot{\alpha}\dot{z}^2)e^{2\alpha(r,t)} + 2\omega^2 z\dot{z} &\Rightarrow \quad \frac{d}{dt}\left(\dot{z}^2 e^{2\alpha(r,t)} + \omega^2 z^2\right) = 0 \\ \Rightarrow \quad \dot{z}^2 e^{2\alpha(r,t)} + \omega^2 z^2 &= \mathcal{J}_{zz} \end{aligned} \quad (2.3.13)$$

Applying the Bohlin map and re-parametrization through

$$\dot{z} = \frac{1}{2}\frac{|\xi|}{(\xi)^{\frac{1}{2}}}\xi' = \frac{1}{2}(\bar{\xi})^{\frac{1}{2}}\xi' \quad (2.3.14)$$

gives us:

$$\mathcal{J}_{zz} = \frac{1}{4}\bar{\xi}(\xi')^2 e^{2\bar{\alpha}(R,t)} + \omega^2 \xi \quad (2.3.15)$$

Isotropy in a system implies that the equation of motion takes the same form along any direction axis. We can use the system isotropy to infer the conjugate equation from (2.3.12):

$$\ddot{\bar{z}} + \dot{\alpha}\dot{\bar{z}} + \omega^2 e^{-2\alpha(r,t)}\bar{z} = 0 \quad (2.3.16)$$

Thus, using (2.3.12) and (2.3.16) by  $\dot{z} \times (2.3.12) + (2.3.16) \times \dot{z}$  we get:

$$\begin{aligned} \{(\dot{z}\ddot{z} + \dot{z}\ddot{\bar{z}}) + 2\dot{\alpha}|\dot{z}|^2\} + \omega^2 e^{-2\alpha(r,t)}(\dot{z}z + \bar{z}\dot{z}) &= 0 \quad \Rightarrow \quad \frac{d}{dt}\left(|\dot{z}|^2 e^{2\alpha(r,t)} + \omega^2 |z|^2\right) = 0 \\ \therefore \quad |\dot{z}|^2 e^{2\alpha(r,t)} + \omega^2 |z|^2 &= \mathcal{J}_{z\bar{z}} \end{aligned} \quad (2.3.17)$$

The Bohlin map transforms (2.3.17) into:

$$\mathcal{J}_{z\bar{z}} = \left( \frac{1}{4} |\xi'|^2 e^{2\tilde{\alpha}(R,t)} + \omega^2 \right) |\xi| \quad \omega^2 = \frac{\mathcal{J}_{z\bar{z}}}{|\xi|} - \frac{1}{4} |\xi'|^2 e^{2\tilde{\alpha}(R,t)} \quad (2.3.18)$$

Thus, using (2.3.15) and (2.3.18) gives us the result:

$$\therefore \quad \mathcal{J}_{zz} = \frac{e^{2\tilde{\alpha}(R,t)}}{4} \left( \bar{\xi} \xi' - \bar{\xi}' \xi \right) \xi' + \mathcal{J}_{z\bar{z}} \frac{\xi}{|\xi|} \quad (2.3.19)$$

Comparing (2.3.19) to

$$\mathbb{A} = -iL(m\xi') - 4m^2 \mathcal{J}_{z\bar{z}} \frac{\xi}{|\xi|} \quad \left( \begin{array}{l} L = ml \\ \tilde{k} = 4m\mathcal{J}_{z\bar{z}} \end{array} \right)$$

we get the equivalent Runge-Lenz vector

$$\mathbb{A} = -iL(m\xi') e^{2\tilde{\alpha}(R,t)} - 4m^2 \mathcal{J}_{z\bar{z}} \frac{\xi}{|\xi|} \quad (2.3.20)$$

This system, appears to be a re-parameterized version of the original harmonic oscillator. Aside from the exponential factor, the form of the equivalent Runge-Lenz vector is the same.

### 2.3.3 Damped Hamiltonian mechanics

The classical lagrangian for a dissipative system according to (2.3.8) can be given by:

$$L = e^{\alpha(\vec{x},t)} \left( \frac{m}{2} |\dot{\vec{x}}|^2 - U \right) \quad (2.3.21)$$

The Hamiltonian by Legendre transformation of (2.3.21) is:

$$H = \frac{1}{2m} |\vec{p}|^2 e^{-\alpha(\vec{x},t)} + U(\vec{x}) e^{\alpha(\vec{x},t)} \quad \vec{p} = m e^{\alpha(\vec{x},t)} \dot{\vec{x}} \quad (2.3.22)$$

The Hamilton's equations of motion are given by:

$$\dot{\vec{x}} = \frac{\partial H}{\partial \vec{p}} = \frac{\vec{p}}{m} e^{-\alpha(\vec{x},t)} \quad \dot{\vec{p}} = -\frac{\partial H}{\partial \vec{x}} = -\vec{\nabla} U e^{\alpha(\vec{x},t)} \quad (2.3.23)$$

Thus, we will find that the Hamiltonian (2.3.22) is dissipative:

$$\frac{dH}{dt} = \left( \dot{\vec{p}} \cdot \frac{\vec{p}}{m} e^{-\alpha(\vec{x},t)} + \dot{\vec{x}} \cdot \vec{\nabla} U e^{\alpha(\vec{x},t)} \right) - \dot{\alpha} \left( \frac{1}{2m} |\vec{p}|^2 e^{-\alpha(\vec{x},t)} - U(\vec{x}) e^{\alpha(\vec{x},t)} \right) = -L\dot{\alpha}$$

This concludes our optical analysis of mechanics and spacetime with drag included. As we can see, the central force, and consequently the potential are time dependent. Now we shall elaborate on the formulation of dynamics related to null geodesics.

### 3 Basic dynamical formulation of null geodesics

In Sec. 2, we deduced optical mechanics from null geodesics. Here, we shall deduce Binet's equation to from null geodesics and dynamically compare them to oscillators. A spherically symmetric Lorentzian  $(n+1)$ -dimensional metric with  $S^{n-1}$  symmetry that is asymptotically flat can be written as:

$$ds^2 = f(r)c^2dt^2 - \frac{dr^2}{g(r)} - r^2d\Omega_{n-1}^2 \quad (3.1)$$

where to account for asymptotic flatness, we write:

$$\begin{aligned} \lim_{r \rightarrow \infty} f(r) = \lim_{r \rightarrow \infty} g(r) = 1 & \Rightarrow f(r) = 1 + F(r), \quad g(r) = 1 + G(r) \\ \text{where } \lim_{r \rightarrow \infty} F(r) = \lim_{r \rightarrow \infty} G(r) = 0 & \Rightarrow F(r) = \sum_{i=2}^{\infty} \frac{a_i}{r^i} \quad G(r) = \sum_{i=2}^{\infty} \frac{b_i}{r^i} \end{aligned} \quad (3.2)$$

The Lagrangian for this metric in a plane is:

$$L = -\frac{m}{2} \left( \frac{ds}{d\tau} \right)^2 = \frac{m}{2} \left( \frac{\dot{r}^2}{g(r)} + r^2\dot{\phi}^2 - f(r)c^2\dot{t}^2 \right) \quad (3.3)$$

We should keep in mind that from 3.3, we can deduce 2 conserved quantities:

$$q = \frac{\partial L}{\partial \dot{t}} = -f(r)c\dot{t} \quad l = \frac{\partial L}{\partial \dot{\phi}} = r^2\dot{\phi} \quad (3.4)$$

The null geodesic is characterised by setting  $ds^2 = 0 \Rightarrow L = 0$  for (3.1) and (3.3). To provide the same formulation employed in Sec. 2, we will define two null geodesics under constraints since the space is not isotropic to define the directional refractive indices according to (2.9).

$$\begin{aligned} \phi = \text{constant} \quad \frac{\dot{r}^2}{g(r)} = f(r)c^2\dot{t}^2 & \Rightarrow n_r^2 = \frac{1}{f(r)g(r)} \\ r = \text{constant} \quad r^2\dot{\phi}^2 = f(r)c^2\dot{t}^2 & \Rightarrow n_\phi^2 = \frac{1}{f(r)} \end{aligned} \quad (3.5)$$

Upon restriction to motion in the plane  $\dot{\theta} = 0$  due to conserved quantities (3.4), the radial co-ordinate inversion  $r \rightarrow u = \frac{1}{r}$  is gives the differential equation:

$$\begin{aligned} \frac{1}{g(r)} \left( \frac{\dot{r}}{\dot{\phi}} \right)^2 + r^2 - \frac{f(r)c^2\dot{t}^2}{\dot{\phi}^2} = 0 & \Rightarrow \left( \frac{1}{r^2} \frac{dr}{d\phi} \right)^2 + \frac{g(r)}{r^2} - g(r) \frac{f(r)c^2\dot{t}^2}{(r^2\dot{\phi})^2} = 0 \\ \therefore \left( \frac{du}{d\phi} \right)^2 + \tilde{g}(u)u^2 = \left( \frac{q}{l} \right)^2 \frac{\tilde{g}(u)}{\tilde{f}(u)} \end{aligned} \quad (3.6)$$

In terms of (3.5), we can re-write (3.6) as:

$$\therefore \left( \frac{du}{d\phi} \right)^2 + \left( \frac{\tilde{n}_\phi}{\tilde{n}_r} \right)^2 u^2 = \left( \frac{q}{l} \right)^2 \left( \frac{\tilde{n}_\phi^2}{\tilde{n}_r} \right)^2 \quad (3.7)$$

Using (3.2), and writing  $b = \frac{l}{q}$ , we can write (3.6) as:

$$\left( \frac{du}{d\phi} \right)^2 + u^2 = \left[ 1 - \left( \frac{\tilde{n}_\phi}{\tilde{n}_r} \right)^2 \right] u^2 + \frac{1}{b^2} \left( \frac{\tilde{n}_\phi^2}{\tilde{n}_r} \right)^2 \quad (3.8)$$

The corresponding Binet's equation can be deduced by differentiating the above equation:

$$\frac{d^2u}{d\phi^2} + u = \frac{F'(u)}{2} \quad F(u) = \frac{1}{b^2} \left( \frac{\tilde{n}_\phi^2}{\tilde{n}_r} \right)^2 - \left[ 1 - \left( \frac{\tilde{n}_\phi}{\tilde{n}_r} \right)^2 \right] u^2 \quad (3.9)$$

If we choose our co-ordinates such the spatial part is conformally flat like (2.1), ie.

$$ds^2 = f(r)c^2dt^2 - \frac{1}{g(r)}(dr^2 + r^2d\Omega_{n-1}^2)$$

$$n_r^2 = n_\phi^2 = \frac{1}{f(r)g(r)} = n^2$$

then we will instead get the equation equivalent to (3.8) and (3.9) as:

$$\left( \frac{du}{d\phi} \right)^2 + u^2 = \left( \frac{n(u)}{b} \right)^2 \quad (3.10)$$

$$\frac{d^2u}{d\phi^2} + u = \frac{n(u)}{b^2} \frac{n'(u)}{n(u)} \quad (3.11)$$

Now we shall look at a few solutions and their related equations.

### 3.1 Solutions and Schwarzschild Tangherlini metric

If we choose the following settings for the coefficient functions of (3.2):

$$\begin{aligned} G(r) = \frac{A}{r^{n-2}} \quad \Rightarrow \quad \tilde{G}(u) = Au^{n-2} \\ \frac{g(r)}{f(r)} = \frac{\tilde{g}(u)}{\tilde{f}(u)} = Bu^n + C \quad \Rightarrow \quad \tilde{f}(u) = \frac{1 + Au^{n-2}}{Bu^n + C} \\ \lim_{u \rightarrow 0} \tilde{f}(u) = 1 \quad \Rightarrow \quad C = 1 \quad \Rightarrow \quad \tilde{f}(u) = \frac{1 + Au^{n-2}}{1 + Bu^n} \end{aligned} \quad (3.1.1)$$

The differential equation (3.8) and (3.9) will become:

$$\left( \frac{du}{d\phi} \right)^2 + u^2 = 2M_n u^n + \frac{1}{b^2} \quad 2M_n = -\left( A + \frac{B}{b^2} \right), \quad (3.1.2)$$

$$\frac{d^2u}{d\phi^2} + u = nM_n u^{n-1} \quad (3.1.3)$$

If  $B = 0$ , we will have  $2M_n = -A$ , meaning that according to (3.5) and (3.1.1):

$$\begin{aligned} f(r) = g(r) = 1 - \frac{2M_n}{r^{n-2}} \\ n_r(r) = (n_\phi(r))^2 = \frac{1}{f(r)} = \left( 1 - \frac{2M_n}{r^{n-2}} \right)^{-1} \end{aligned} \quad (3.1.4)$$

which results in the Schwarzschild-Tangherlini solution:

$$ds^2 = -\left( 1 - \frac{2M_n}{r^{n-2}} \right) dt^2 + \frac{dr^2}{1 - \frac{2M_n}{r^{n-2}}} + r^2 d\Omega_{n-2}^2 \quad (3.1.5)$$

Now we will look at two other solutions of Einstein's equations.

### 3.2 Other metrics

Most solutions of Einstein's equations for spherically symmetric spaces will have  $f(r) = g(r)$ , where we will have two such examples:

$$f_H(r) = 1 - \Lambda r^2 - \frac{2M}{r} = 1 - \frac{\Lambda}{u^2} - 2Mu \quad (3.2.1)$$

$$f_{HD}(r) = 1 - \Lambda r^2 - \frac{Q}{r^2} - \frac{2M}{r} = 1 - \frac{\Lambda}{u^2} - Qu^2 - 2Mu \quad (3.2.2)$$

The refractive indices of these spaces are given as they were in (3.1.4):

$$n_{rH}(r) = (n_{\phi H}(r))^2 = \left(1 - \frac{\Lambda}{u^2} - 2Mu\right)^{-1} \quad (3.2.3)$$

$$n_{rHD}(r) = (n_{\phi HD}(r))^2 = \left(1 - \frac{\Lambda}{u^2} - Qu^2 - 2Mu\right)^{-1} \quad (3.2.4)$$

The related corresponding Binet's equations are given respectively as

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 \quad (3.2.5)$$

$$\frac{d^2u}{d\phi^2} + u = 3Mu^2 + 2Qu^3 \quad (3.2.6)$$

The above 2 results are equivalent to the equations for the Helmholtz oscillator [21], and the Helmholtz-Duffing oscillator [22] respectively, both of which are nonlinear equations that have received a lot of attention recently for the wide range of applications in engineering. The solution of the Helmholtz equation (3.2.5) is given in terms of Jacobi elliptic function  $sn$ . The exact solution of the Helmholtz-Duffing oscillator equation (3.2.6) can also be expressed in terms of Jacobi elliptic function [23, 24]. It must be worth to note that Gibbons and Vyska [8] used Weierstrass elliptic functions to give a full description and classification of null geodesics in Schwarzschild spacetime.

### 3.3 Equation of motion

Null geodesics of (3.1) may be mapped into non-relativistic central-force motion described by:

$$\begin{aligned} \dot{r}^2 + l^2 \frac{g(r)}{r^2} &= q^2 \frac{g(r)}{f(r)} \quad \Rightarrow \quad 2\ddot{r} + l^2 \left( \frac{1+G(r)}{r^2} \right)' = q^2 \left( \frac{g(r)}{f(r)} \right)' \\ \therefore \quad \ddot{r} - r\dot{\phi}^2 &= -l^2 \left( \frac{G(r)}{r^2} \right)' + q^2 \left( \frac{g(r)}{f(r)} \right)' \end{aligned} \quad (3.3.1)$$

in a central force given by:

$$\begin{aligned} F(r) &= -l^2 \left( \frac{G(r)}{r^2} \right)' + q^2 \left( \frac{g(r)}{f(r)} \right)' \\ V(r) &= l^2 \left( \frac{G(r)}{r^2} \right) - q^2 \left( \frac{g(r)}{f(r)} \right) \end{aligned} \quad (3.3.2)$$

This shows that the central force and potential for solutions to the Einstein's equations like the Schwarzschild-Tangherlini, Helmholtz, and Helmholtz-Duffing systems are:

$$F_{ST}(r) = -\frac{2nl^2M_n}{r^{n+1}} \quad V_{ST}(r) = -l^2\left(\frac{2M_n}{r^n}\right) - q^2 \quad (3.3.3)$$

$$F_H(r) = -\frac{6l^2m}{r^4} \quad V_H(r) = -l^2\left[\Lambda + \left(\frac{2m}{r^3}\right)\right] - q^2 \quad (3.3.4)$$

$$F_{HD}(r) = -l^2\left[\frac{6m}{r^4} + \frac{4Q}{r^5}\right] \quad V_{HD}(r) = -l^2\left[\Lambda + \left(\frac{2m}{r^3}\right) + \frac{Q}{r^4}\right] - q^2 \quad (3.3.5)$$

We shall next elaborate on how to describe unparametrised geodesics and the usage of isotropic co-ordinates.

### 3.4 Unparameterised geodesics and isotropic co-ordinates

Since we are dealing with null geodesics, the geodesic cannot be parametrised along a vanishing curve. However, we have seen that the geodesics can be produced by extremising the spatial curve as shown in (2.1.1) to take the least time to traverse in accordance with Fermat's principle, with respect to which it can be parametrised.

By writing  $ds_{\mathcal{O}}^2 = dt^2$  in (3.6), we get the metric:

$$ds_{\mathcal{O}}^2 = \frac{dr^2}{f(r)g(r)} + \frac{r^2}{f(r)}d\phi^2 \quad (3.4.1)$$

which describes the unparameterised geodesics of the optical 2-metric. If we choose to write it in isotropic co-ordinates as follows, we get:

$$ds_{\mathcal{O}}^2 = \eta^2(\rho)(d\rho^2 + \rho^2d\phi^2)$$

$$\eta(\rho) d\rho = \frac{dr}{\sqrt{f(r)g(r)}} \quad \eta(\rho)\rho = \frac{r}{\sqrt{f(r)}} \quad \Rightarrow \quad \frac{d\rho}{\rho} = \frac{dr}{r\sqrt{g(r)}}$$

In case of Schwarzschild solution (3.1.5) with ( $n = 3$ ), the isotropic co-ordinate is given by:

$$\frac{d\rho}{\rho} = \frac{dr}{\sqrt{r(r-2M_3)}} \equiv \frac{dx}{\sqrt{(x^2 - M_3^2)}} = d\left[\ln\left\{\frac{x}{M_3} + \sqrt{\left(\frac{x}{M_3}\right)^2 - 1}\right\}\right] \quad r = x + M_3$$

$$\rho = c\left(\frac{r}{M_3} - 1 + \frac{1}{M_3}\sqrt{r(r-2M_n)}\right)$$

where, for  $\rho = M_3$  for  $r_0 = 2M_n$ , we have  $c = M_3$ . Thus, the isotropic co-ordinate is

$$\therefore \quad \rho = r - M_3 + \sqrt{r(r-2M_3)} \quad (3.4.2)$$

for which  $\eta(\rho)$  is given by

$$\eta(\rho) = \frac{(\rho + M_3)^3}{\rho^2(\rho - M_3)} \quad (3.4.3)$$

We will next examine how null geodesics exhibit dualities under conformal transformations.



## 4 Duality under conformal transformation

A conformal transformation that preserves the Jacobi metric will reveal potential power law dualities. If we use complex variables to describe the planar co-ordinates as  $z = x + iy$ , then we can describe the Jacobi metric as:

$$ds_J^2 = 2m(E - V(|z|))d\bar{z} dz \quad (4.1)$$

If we employ the pullback with the conformal map:

$$z \longrightarrow w = w(z) \quad (4.2)$$

Then we will get the complex system which is a projective dual [25] of (4.1):

$$\begin{aligned} ds^2 &= 2m(\tilde{E} - \tilde{V}(|w|))d\bar{w} dw \\ \tilde{E} &= V(|z|)|w'(z)|^{-2} \quad \tilde{V}(|w|) = E|w'(z)|^{-2} \end{aligned} \quad (4.3)$$

Let us consider only conformal maps of the form  $w = z^p$ . This means that according to (4.5) the potential has to be :

$$\begin{aligned} V \propto |z|^a \quad |w'(z)| \propto z^{p-1} \quad \Rightarrow \quad a &= 2(p-1) \\ V(|z|) \propto |z|^{2p-2} \end{aligned}$$

Conversely, this means that

$$\tilde{V}(|w|) \propto |z|^{2-2p} = |w|^{\frac{2-2p}{p}}$$

where, for various settings of  $p$ , we will get various dualities that preserve the form of the non-relativistic Jacobi metric:

1. For  $p = 2$ , we essentially get the Kepler-Hooke duality.

$$V(|z|) \propto |z|^2 \quad \tilde{V}(|w|) = |w|^{-1}$$

also known as the Bohlin-Arnold duality, an equivalence between the Kepler and Hooke mechanical systems in the plane originating in a paper by Bohlin [26], and Arnold [27]. Let us apply this co-ordinate conversion as  $w : (r, \phi) \longrightarrow (y = r^2, \varphi = 2\phi)$ . This should let us write from the null version of lagrangian (3.3):

$$\begin{aligned} -\frac{f(r)\dot{t}^2}{\dot{\phi}^2} + \frac{1}{g(r)}\left(\frac{\dot{r}}{\dot{\phi}}\right)^2 + r^2 &= 0 \quad \Rightarrow \quad \left(\frac{dr}{d\phi}\right)^2 + g(r)r^2 - \frac{1}{b^2}\frac{g(r)}{f(r)}r^4 = 0 \\ \Rightarrow \quad -g(r)\frac{f(r)(r^2)^3\dot{t}^2}{(r^2\dot{\phi})^2} + \left(r\frac{dr}{d\phi}\right)^2 + g(r)(r^2)^2 &= 0 \\ \Rightarrow \quad \left(\frac{dy}{d\varphi}\right)^2 + y^2 &= y^3\frac{\tilde{g}(y)}{\tilde{f}(y)}\frac{q^2}{l^2} - y^2\tilde{G}(y) \end{aligned}$$

where for  $f(r) = g(r)$  and  $G(r) \propto r^{2-n} = y^{1-\frac{n}{2}}$  and  $A = 2M_n$ ,  $B = b^{-2}$  for the Schwarzschild-Tangherlini metric (3.1.5), the equation in original co-ordinates is:

$$\left(\frac{dr}{d\phi}\right)^2 + r^2 = Ar^{4-n} + Br^4 \quad (4.4)$$

and, under conformal transformation  $w : (r, \phi) \longrightarrow (y = r^2, \varphi = 2\phi)$ , we get:

$$\left(\frac{dy}{d\varphi}\right)^2 + y^2 = Ay^3 + By^{3-\frac{n}{2}}$$

Upon setting  $n = 6$  in the above result, we will see that (4.4) is dual to

$$\left(\frac{dy}{d\varphi}\right)^2 + y^2 = Ay^3 + B$$

which is the equation (3.1.2) with  $n = 3$ .

2. For  $p = -1$ , we get self-duality.

$$\tilde{V}(|w|) \sim V(|z|) \propto |z|^{-4} = |w|^{-4}$$

Now, looking at (3.1.2), for the co-ordinate transformation  $u = r^{-1}$ , we see that (4.4) is dual to

$$\left(\frac{du}{d\psi}\right)^2 + u^2 = Au^n + B$$

The dual versions of the equations above are identical in form only for  $n = 4$ , which according to (3.3.2), for  $f(r) = g(r)$  and  $G(r) \propto r^{2-n}$  means that  $V(r) \propto r^{-4}$ . This shows that null geodesics for Schwarzschild-Tangherlini metrics for  $n = 4$  exhibit self-dual orbits under co-ordinate inversion  $(r, \phi) \leftrightarrow (u = r^{-1}, \psi = -\phi)$ .

3. For  $p = -\frac{1}{2}$ , we get what will be the focus of our discussion.

$$V(|z|) \propto |z|^{-3} \quad \tilde{V}(|w|) = |w|^{-6}$$

This conformal map is essentially the combination of the above two conformal maps as we shall describe below. If we write the conformal map  $w = z^{-\frac{1}{2}}$ , it essentially means

$$z = \frac{1}{w^2}$$

Here, we shall compare the equations for conformal transformations ( $p = 2, n = 6$ ) and ( $p = -1, n = 3$ ). Let us define a co-ordinate  $u = r^{-1}$ , such that we have the conformal co-ordinate map

$$(y, \varphi) \longrightarrow \left(\frac{1}{u^2}, -2\psi\right) \quad (4.5)$$

to (3.1.2) for the case of  $n = 3$ . This will give us:

$$\left(\frac{dy}{d\varphi}\right)^2 + y^2 = 2M_n y^3 + \frac{1}{b^2}$$

which under the conformal map (4.5) transforms into

$$\left(\frac{du}{d\psi}\right)^2 + u^2 = 2M_n + \frac{u^6}{b^2} = 2\widetilde{M}_n u^6 + \frac{1}{\widetilde{b}^2}$$

Showing that the cases  $n = 3$  and  $n = 6$  are dual to each other, provided we redefine the co-efficients as  $2\widetilde{M}_n = \frac{1}{b^2}$ ,  $\widetilde{b}^2 = \frac{1}{2M_n}$ .

Thus, null-geodesics derived from the Schwarzschild-Tangherlini metric under Bohlin transformation for  $n = 6$ , and under co-ordinate inversion for  $n = 3$  produce dual equations.

## 5 Conclusion and Discussion

We managed to describe spacetimes with gravitational fields as optically refractive media, and elaborately reformulated mechanics in the classical limit in an optical-mechanical form. However, we have shown that the Hamilton-Jacobi equation for optical mechanical formulation was not comparable to the Eikonal equation as claimed in [13, 19, 20]. We have also revealed that isotropic spaces describe mechanics with a drag force included, which means that such mechanics can be recast and analyzed in the form of the Gorriange Leach equations.

It was shown that null-geodesics mechanics can be compared to a central force mechanical system via Binet's equation, and for two choices of metric co-efficient functions, we can get systems with drag like Helmholtz [21] and Helmholtz-Duffing [22] oscillators. Dual systems that preserve the classic Jacobi metric under conformal transformation were explored, one of which was the Bertrand system pair.

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